Limits to error correction in quantum chaos

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(December 2000)

We study the correction of errors that have accumulated in an entangled state of spins as a result of unknown local variations in the Zeeman energy (B) and spin-spin interaction energy (J). A non-degenerate code with error rate κ can recover the original state with high fidelity within a time $t_{\rm R} \simeq \hbar\kappa^{1/2}/\max(B,J)$ — independent of the number of encoded qubits. Whether the Hamiltonian is chaotic or not does not affect this time scale, but it does affect the complexity of the error-correcting code.

PACS numbers: 03.67.Lx, 05.45.Mt, 24.60.Lz, 74.40.+k

In classical mechanics, chaos severely limits the operation of a reversible computer [1]. Any uncertainty in the initial conditions is magnified exponentially by chaotic dynamics, rendering the outcome of the computation unpredictable. This is why practical computational schemes are irreversible: Dissipation suppresses chaos and makes the computation robust to errors [2]. A quantum computer does not have this option; It relies on the reversible unitary evolution of entangled quantum mechanical states, which does not tolerate dissipation [3]. This invites the question [4,5], what limitations quantum chaos might pose on quantum computing.

To answer this question one needs to consider the possibilities and restrictions of quantum error correction [6]. Errors can occur due to interaction with the environment (errors of decoherence) and due to uncertainty in the unitary evolution (unitary errors). The original state can be recovered reliably if the errors involve at most a fraction $\kappa \lesssim 0.1$ of the total number of qubits. The corresponding maximal time during which errors may be allowed to accumulate (the recovery time $t_{\rm R}$) is easy to find if different qubits are affected independently. That may be a reasonable assumption for certain mechanisms of decoherence and also for unitary errors resulting from an uncertain single-particle Hamiltonian. Uncertainties in the interactions among the qubits pose a more complex problem [7].

Georgeot and Shepelyansky [4] studied this problem for a model Hamiltonian of N interacting spins that exhibits a transition from regular dynamics (nearly isolated spins) to chaotic dynamics (strongly coupled spins). They concluded for the chaotic regime that $t_{\rm R}$ goes to zero $\propto 1/N$ for large N, but their analysis did not incorporate the optimal error-correcting procedure. We assume a good (non-degenerate) error-correcting code and obtain a recovery time of the order of the inverse energy uncertainty per spin — irrespective of the number of encoded qubits. By considering both phase-shift and spin-flip errors we find that $t_{\rm R}$ is insensitive to whether the Hamiltonian is chaotic or not. (The authors of Refs. [4,5] arrived at the opposite conclusion that $t_{\rm R}$ increases strongly when chaos is suppressed, but they took only spin-flip errors into account.) The absence of chaos can be used to reduce the complexity of the code, in that a *classical* error-correcting code suffices for the majority of the errors in the regime of regular dynamics.

The Hamiltonian H under consideration describes N coupled spins $\boldsymbol{\sigma}_n = (\sigma_n^x, \sigma_n^y, \sigma_n^z)$ on a lattice in a magnetic field $\mathbf{B}_n = (B_n^x, B_n^y, B_n^z)$,

$$H = \sum_n \mathbf{B}_n \cdot \boldsymbol{\sigma}_n + \sum_{n\neq m} \boldsymbol{\sigma}_n \cdot \mathbf{J}_{nm} \cdot \boldsymbol{\sigma}_m. \tag{1}$$

A spin n interacts with d neigboring spins m via the matrix $\mathbf{J}_{nm}$. The spin could be a nuclear spin or the spin of an electron confined to a quantum dot, in the context of solid-state based proposals for quantum computing [8–10]. More generally, the spin could be a representation of a two-level system (for example, in the context of the ion-trap quantum computer [11]). We assume that $\mathbf{B}_n$ and $\mathbf{J}_{nm}$ fluctuate independently from site to site, with zero mean and variance $\overline{|\mathbf{B}_n|^2} = B^2$ and $\sum_{\alpha\beta} \overline{(J_{nm}^{\alpha\beta})^2} = J^2$ (provided m is one of the d neighbors of n, otherwise $\mathbf{J}_{nm} = 0$). We denote by $U = (B^2 + 2dJ^2)^{1/2}$ the root-mean-square energy uncertainty per spin.

A state ψ_0 evolves in time according to $\psi(t) = e^{-iHt}\psi_0$ (setting $\hbar \equiv 1$). We assume that we do not know the parameters of the Hamiltonian, and use quantum error correction to recover ψ_0 from $\psi(t)$ [12]. Let ψ_0 lie in the code space of a non-degenerate error-correcting code [6]. The code space is a 2^M dimensional subspace of the full 2^N dimensional Hilbert space, such that

$$\langle\psi_0|\sigma_{n_1}^{\alpha_1}\sigma_{n_2}^{\alpha_2}\ldots\sigma_{n_k}^{\alpha_k}|\psi_0'\rangle = 0, \;\; 1 \le k \le 2K, \tag{2}$$

for any two (possibly identical) states ψ_0, ψ_0' in the code space and any product of up to $2K$ Pauli matrices σ_n^α (acting on different spins $n_1, n_2, \ldots$). The number M is the number of qubits encoded in N spins. The number K is the number of errors that the code can correct, where the application of σ_n^x, σ_n^y, or σ_n^z to any of the N spins counts as one error. The ratio $M/N = \rho$ is the bit rate of the code and the ratio $K/N = \kappa$ the error rate.

Error correction is successful if $\psi(t)$ lies in the error space of ψ_0, which is the subspace spanned by the state ψ_0 and the states derived from ψ_0 by making up to K errors. The operator $\mathcal{P}$ projects onto the error space. Explicitly: $\mathcal{P} = \sum_{p=0}^{K} \mathcal{P}_p$, with

$$\mathcal{P}_p = \sum_{\{n,\alpha\}} \frac{1}{p!} \sigma_{n_1}^{\alpha_1} \dots \sigma_{n_p}^{\alpha_p} |\psi_0\rangle\langle\psi_0| \sigma_{n_1}^{\alpha_1} \dots \sigma_{n_p}^{\alpha_p}. \tag{3}$$

The symbol $\sum_{\{n,\alpha\}}$ indicates a summation over the n_i's and α_i's, with the restriction that the indices $n_1, n_2 \dots$ should be all distinct. (The indices $\alpha_1, \alpha_2 \dots$ need not be distinct.) The norm

$$F(t) = |\mathcal{P}\psi(t)|^2 = \langle\psi_0| e^{iHt} \mathcal{P} e^{-iHt} |\psi_0\rangle \tag{4}$$

of the projected state is the probability of successful error correction after a time t. It is the "fidelity" of the recovered state [6]. The recovery time $t_{\rm R}$ can be defined as the time at which the fidelity has dropped from 1 to 1/2.

We assume that the error-correcting code is "good", meaning that ρ and κ tend to a non-zero value as $N \to \infty$. Good quantum-error correcting codes exist, but their construction for large N is a complex problem [13–18]. Our strategy will be to derive a lower bound to F and $t_{\rm R}$ that does not use any properties of the code beyond the non-degeneracy condition (2), so that we can avoid an explicit construction. An alternative approach would be to abandon the requirement of a "good" code, and keep the number M of encoded qubits fixed as the total number of spins N goes to infinity. One can then use the technique of concatenation [6] to construct codes that are safe for a large number of errors at the expense of a vanishingly small bit rate ρ. (See Ref. [19] for such a calculation in the case $M = 1$.)

Our first step is to decompose the evolution operator $e^{iHt} = \sum_{k=0}^{N} X_k$ into operators X_k that create k errors. For $k \ll N$ and $t \ll 1/U$ we may approximate

$$X_k = X_0 \sum_{q=0,2,4}^{k} \sum_{\{n,\alpha\}} \frac{(it)^{k-q/2}}{(k-q)!(q/2)!} \sigma_{n_1}^{\alpha_1} \dots \sigma_{n_k}^{\alpha_k} \times J_{n_1 n_2}^{\alpha_1 \alpha_2} \dots J_{n_{q-1} n_q}^{\alpha_{q-1} \alpha_q} B_{n_{q+1}}^{\alpha_{q+1}} \dots B_{n_k}^{\alpha_k}, \tag{5}$$

$$X_0 = \exp\left[-\tfrac{1}{2} t^2 \sum_n |\mathbf{B}_n|^2 - t^2 \sum_{n \neq m} \sum_{\alpha,\beta} (J_{nm}^{\alpha\beta})^2 \right]. \tag{6}$$

The approximation consists in neglecting terms in the exponent of order $k(Ut)^2$ and $N(Ut)^4$, relative to the terms retained of order $N(Ut)^2$. We may write $X_0 \approx \exp[-\frac{1}{2} N (Ut)^2]$, neglecting fluctuations in the exponent that are smaller by a factor $1/\sqrt{N}$.

We next substitute the decomposition of e^{iHt} in error operators into the fidelity (4),

$$F(t) = \sum_{p=0}^{K} \sum_{k,k'=0}^{N} \langle X_k \mathcal{P}_p X_{k'}^* \rangle, \tag{7}$$

where we have abbreviated $\langle \cdots \rangle = \langle\psi_0| \cdots |\psi_0\rangle$. To simplify this expression, we take the average over the random variations in the B_n's and J_{nm}'s. (We will show later that statistical fluctuations around the average are insignificant.) Only the terms with $k = k'$ contribute to the average. The terms with $p + k \leq 2K$ can be simplified further, since they contain at most $2K$ Pauli matrices. In view of Eq. (2), these expectation values vanish unless the product of Pauli matrices reduces to a c-number, which requires $p = k$. Hence the average fidelity can be written as $\bar{F} = F_1 + F_2$, with

$$F_1 = \sum_{p=0}^{K} \sum_{\{n,\alpha\}} \frac{1}{p!} \overline{|\langle X_p \sigma_{n_1}^{\alpha_1} \dots \sigma_{n_p}^{\alpha_p} \rangle|^2}, \tag{8}$$

$$F_2 = \sum_{p=0}^{K} \sum_{k=2K+1-p}^{N} \sum_{\{n,\alpha\}} \frac{1}{p!} \overline{|\langle X_k \sigma_{n_1}^{\alpha_1} \dots \sigma_{n_p}^{\alpha_p} \rangle|^2}. \tag{9}$$

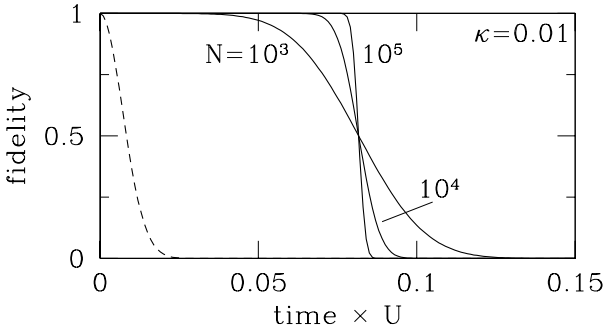

FIG. 1. Solid curves: time dependence of the lower bound F_1 to the ensemble-averaged fidelity, calculated from Eq. (11) for error rate $\kappa = 0.01$ and three values of N. (We took $B^2 = 2dJ^2$, so that the root-mean-squared energy uncertainty per spin U is equally divided between Zeeman energy and interaction energy.) The dashed curve shows (for $N = 10^4$) the squared overlap $X_0^2 = \exp[-N(Ut)^2]$ between initial and final state. In all these curves the number M of encoded qubits is a fixed fraction ρ of the total number of spins N.

The expectation values in F_1 are evaluated by substituting Eq. (5) and extracting the terms that reduce to a c-number,

$$F_1 = e^{-N(Ut)^2} \sum_{p=0}^{K} \sum_{q=0,2,4}^{p} \frac{(NB^2t^2)^{p-q}}{(p-q)!} \frac{(2NdJ^2t^2)^{q/2}}{(q/2)!}. \tag{10}$$

For $K \gg 1$ we may approximate $e^{-x} x^k/k! \approx (2\pi x)^{-1/2} \exp[-(k-x)^2/2x]$ and replace the sums in Eq. (10) by integrals. The result is

$$F_1 = \tfrac{1}{2} + \tfrac{1}{2} \mathrm{erf}[(t_{\rm R} - t)/\Delta t], \tag{11a}$$

$$t_{\rm R} = \sqrt{\frac{\kappa}{B^2 + 4dJ^2}}, \quad \Delta t = \sqrt{\frac{1}{2N}} \frac{\sqrt{B^2 + 8dJ^2}}{B^2 + 4dJ^2}, \tag{11b}$$

with $\mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x e^{-y^2}dy$ the error function. Corrections to $t_{\rm R}$ and Δt arising from the approximations made in Eqs. (5) and (6) are smaller by a factor κ. The expectation values in F_2 depend specifically on ψ_0, hence on the way in which M qubits are encoded in N spins. Since $F_2 \geq 0$ we have a lower bound $\bar{F} \geq F_1$ on the fidelity that is *code independent* within the class of non-degenerate error-correcting codes.

For $N \to \infty$ the time dependence of Eq. (11) approaches the step function $\theta(t_{\rm R} - t)$. The threshold $t_{\rm R}$ is independent of N, while the width Δt of the transition vanishes as $N^{-1/2}$ (solid curves in Fig. 1). These are results for the ensemble-averaged fidelity, but since the variance is bounded by $0 \leq \mathrm{var}\, F \leq \bar{F}(1-\bar{F})$ the fluctuations are insignificant except in the narrow transition region. The step-function behavior of the fidelity also implies that the positive code-dependent term F_2 that we have not included in Fig. 1 satisfies $\lim_{N\to\infty} F_2 \to 0$ for $t < t_{\rm R}$ (since $F_1 + F_2 \leq 1$ and $F_1 \to 1$ for $t < t_{\rm R}$). Any code-dependence of the fidelity can therefore only appear for times greater than $t_{\rm R}$.

The independence of the recovery time on the number M of encoded qubits disagrees with Refs. [4,5]. These authors calculated the squared overlap $|\langle\psi_0|\psi(t)\rangle|^2 \approx X_0^2$ of the time-dependent state with the original state, and argued that the original state would be effectively lost once this overlap is $\ll 1$. However, the original state can be recovered even when this overlap has become exponentially small, if a good error-correcting code is used (compare dashed and solid curves in Fig. 1). The recovery time is increased by a factor $\sqrt{\kappa M}$, with an overhead of $1/\rho$ spins per encoded qubit.

We find that $t_{\rm R}$ at a given U is insensitive to the relative magnitude of B and J, and hence *insensitive to whether the Hamiltonian is chaotic or not.* This conclusion may seem surprising in view of the fact that the eigenstates are completely different in the chaotic and regular regimes [4]: For $J < B/N$ the eigenstates of the total Hamiltonian H are a superposition of a small number of eigenstates of the non-interacting part $\sum_n \mathbf{B}_n\cdot\boldsymbol{\sigma}_n$. This number (known as the participation ratio) increases with increasing J, and when $J \approx B$ it becomes of the same order as the dimension 2^N of the entire Hilbert space. (See Ref. [20] for a description of the onset of quantum chaos in systems with random two-body interactions.) As we will now discuss, the reason that a small participation ratio does not improve the fidelity is that it counts spin-flip errors but not phase-shift errors. For the same reason, suppression of chaos does help to reduce the complexity of the error-correcting code.

The three Pauli matrices correspond to three types of errors: spin flips (σ^x), phase shifts (σ^z), and a combination of the two ($\sigma^y = i\sigma^x\sigma^z$). The complexity of the code is reduced substantially if there is only one type of error to correct. (One can then use a code for classical bits, such as the Hamming code [6].) Suppose that we seek to suppress spin-flip errors, of either type σ^x or σ^y. To this end we impose on the spins a known uniform magnetic field in the z-direction, with Zeeman energy B_0 that is large compared to the magnitude U of the random energy variations. The new Hamiltonian is $H + H_0$, with H given by Eq. (1) and $H_0 = B_0\sum_n \sigma_n^z$. Since B_0 is known we can undo the evolution of a state due to H_0 by applying the operator $e^{iH_0t} = \prod_n(\cos B_0 t + i\sigma_n^z \sin B_0 t)$. Any remaining deviation of $\psi(t)$ from ψ_0 has to be dealt with by the error-correcting code, with projection operator $\mathcal{P}$. The fidelity of the corrected state is $F(t) = |\mathcal{P}G(t)\psi_0|^2$, where the evolution operator G is defined by

$$G(t) = e^{iH_0t}e^{-i(H+H_0)t} = \mathcal{T}\exp\biggl(-i\int_0^t H(t')dt'\biggr). \quad (12)$$

(The notation $\mathcal{T}$ indicates time ordering of the operators $H(t) = e^{iH_0t}He^{-iH_0t}$.)

For $B_0 t \gg 1$ we may replace $H(t)$ by its time average over the interval $(t, t+1/B_0)$. The terms containing a single σ^x or σ^y average out to zero and we are left with

$$G(t) = e^{-it(H_\parallel + H_\perp)}, \quad H_\parallel = \sum_n B_n^z\sigma_n^z + \sum_{n\neq m} J_{nm}^{zz}\sigma_n^z\sigma_m^z,$$

$$H_\perp = \sum_{n\neq m} \tfrac{1}{2}(J_{nm}^{xx} + J_{nm}^{yy})(\sigma_n^x\sigma_m^x + \sigma_n^y\sigma_m^y). \quad (13)$$

(We have assumed $J_{nm}^{xy} = J_{nm}^{yx}$, so that the mixed terms $\sigma_n^x\sigma_m^y$ cancel.) The time dependence of the fidelity is again given by Eq. (11), with $B^2 = \overline{(B_n^z)^2}$ and $J^2 = \overline{(J_{nm}^{zz})^2} + \frac{1}{4}\overline{(J_{nm}^{xx} + J_{nm}^{yy})^2}$. The recovery time $t_{\rm R}$ depends only weakly on the ratio J/B. The relative number of phase-shift and spin-flip errors, however, depends strongly on this ratio. Indeed, if one would use a code that corrects up to $K_\parallel$ errors from σ^z and up to $K_\perp$ errors from σ^x or σ^y, then the maximal $t_{\rm R}$ (at fixed $K_\parallel + K_\perp$) is reached for $K_\perp/K_\parallel = 4dJ^2/B^2$. For $J \ll B$ one has $K_\perp \ll K_\parallel$, so that a classical error-correcting code suffices for the majority of errors.

Before concluding we briefly consider the case that the parameters $\mathbf{B}_n$ and $\mathbf{J}_{nm}$ in the Hamiltonian are not only unknown but also time dependent. The result (10) still holds if we replace $(Bt)^2$ by the correlator $b(t) = \int_0^t dt' \int_0^t dt'' \overline{\mathbf{B}_n(t')\cdot\mathbf{B}_n(t'')}$, and similarly replace $(Jt)^2$ by $j(t) = \int_0^t dt' \int_0^t dt'' \sum_{\alpha\beta} \overline{J_{nm}^{\alpha\beta}(t')J_{nm}^{\alpha\beta}(t'')}$. For a short-time correlation one has $b(t) = b_0|t|$, $j(t) = j_0|t|$. This leads for $K \gg 1$ to the fidelity

$$F_1 = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{erf}[(t_{\rm R} - t)/\Delta t], \quad (14a)$$

$$t_{\rm R} = \frac{\kappa}{b_0 + 4dj_0}, \quad \Delta t = \sqrt{\frac{2\kappa}{N}}\frac{\sqrt{b_0 + 8dj_0}}{(b_0 + 4dj_0)^{3/2}}. \quad (14b)$$

The recovery time now depends linearly on the error rate κ, but it remains N-independent. The next step towards fault-tolerant computing would be to include in the Hamiltonian a part with a known time dependence, representing the logical gates. We leave that for a future investigation.

In conclusion, we have derived a code-independent lower bound for the fidelity F of a state that has been recovered after a unitary evolution for a time t in an unknown random magnetic field and spin-spin interaction. For a large system the transition from $F = 1$ to $F = 0$ occurs abruptly at a time $t_{\rm R}$ that is independent of the total number of spins N and the number of encoded qubits M. The magnitude of $t_{\rm R}$ is set by the inverse energy uncertainty per spin, regardless of whether the spins are nearly isolated or strongly coupled. The suppression of chaos that occurs when the spins are decoupled does not improve the fidelity, because of the persistence of phase-shift errors. Spin-flip errors can be suppressed, and this helps to reduce the complexity of the error-correcting code.

In this work we have concentrated on the recovery from unitary errors. One might question whether suppression of quantum chaos improves the fidelity for recovery from errors of decoherence, in particular in view of the "hypersensitivity to perturbation" observed in computer simulations of systems with a chaotic dynamics [21,22]. This question presents itself as an interesting topic for future research.

We thank P. Zoller for a valuable discussion. This work was supported by the Dutch Science Foundation NWO/FOM. PGS acknowledges the support of the RFBR grant number 98–02–17905.

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